



TITLE:

# On the State Extension and Quantum Correlations for CAR Systems (The structure of operator algebras and its applications)

AUTHOR(S):

Moriya, Hajime

---

CITATION:

Moriya, Hajime. On the State Extension and Quantum Correlations for CAR Systems (The structure of operator algebras and its applications). 数理解析研究所講究録 2003, 1300: 37-51

ISSUE DATE:

2003-01

URL:

<http://hdl.handle.net/2433/42717>

RIGHT:

# CAR系での状態の拡張可能性と量子相関について

## (On the State Extension and Quantum Correlations for CAR Systems)

守屋 創 Hajime Moriya  
高エネ研

### 1 Introduction

A quantum system is described by a  $C^*$ -algebra  $\mathcal{A}$  and its state is given by a normalized positive linear functional  $\varphi$  of  $\mathcal{A}$ . Subsystems of  $\mathcal{A}$  are described by  $C^*$ -subalgebras  $\mathcal{A}(\{i\})$ ,  $i = 1, 2, \dots$ . If the subalgebras  $\mathcal{A}(\{i\})$  generate  $\mathcal{A}$  as a  $C^*$ -algebra, then  $\mathcal{A}$  is called a total system  $\mathcal{A}$ .

Let  $\varphi$  be a state of  $\varphi$ . Then the restrictions of  $\varphi$  to  $\mathcal{A}(\{i\})$  are given by

$$\varphi_i(A) = \varphi(A),$$

for  $A \in \mathcal{A}(\{i\})$ . Each  $\varphi_i$  is a state of  $\mathcal{A}(\{i\})$ .

Conversely, suppose that states  $\varphi_i$  of  $\mathcal{A}(\{i\})$ ,  $i = 1, 2, \dots$ , are first given. If the restriction of the total state  $\varphi$  to  $\mathcal{A}(\{i\})$  is equal to the given state  $\varphi_i$  for each  $i$ , then this state  $\varphi$  is called a joint extension of states  $\varphi_i$  of  $\mathcal{A}(\{i\})$ ,  $i = 1, 2, \dots$ .

For spin lattice or Boson systems, algebras  $\mathcal{A}(\{i\})$  of subsystems with mutually disjoint localization mutually commute and form a tensor product system. Here the total system  $\mathcal{A}$

is generated by the tensor product of  $\mathcal{A}(\{i\})$ ,  $i = 1, 2, \dots$  as follows.

$$\mathcal{A} = \otimes_i \mathcal{A}(\{i\}). \quad (1)$$

Let a set of states  $\varphi_i$  of  $\mathcal{A}(\{i\})$  ( $i = 1, 2, \dots$ ) be given. For tensor product systems, we have obviously a state extension as the tensor product of states  $\varphi_i$ :

$$\varphi = \otimes_i \varphi_i. \quad (2)$$

(In general, there are many state extensions of  $\varphi_i$  other than this product state extension. Note that if all  $\varphi_i$  are pure states, then the joint extension is uniquely given by the product state extension and is a pure state.)

Let us consider the different situations where the subsystems  $\mathcal{A}(\{i\})$  are not commutative for any distinct indices  $i$ . (We assume that intersections of subsystems of disjoint regions do not have non-trivial elements, i.e.,  $\mathcal{A}(\{i\}) \cap \mathcal{A}(\{j\}) = c\mathbf{1}$  ( $c \in \mathbb{C}$ ) for  $i \neq j$ .) Assume that the total system  $\mathcal{A}$  is algebraically generated by  $\mathcal{A}(\{i\})$   $i = 1, 2, \dots$  as

$$\mathcal{A} = \bigvee_i \mathcal{A}(\{i\}). \quad (3)$$

Here there arises the natural question on the state extension from subsystems to the joint system for non-tensor product systems as follows.

Does a state extension of the total system  $\mathcal{A}$  exist for a set of given states  $\varphi_i$  of  $\mathcal{A}(\{i\})$ ? What kind of state extensions are possible or impossible for  $\varphi_i$ ? When is a state extension to be a product state? Is it possible to make a product state extension for given  $\varphi_i$ ?

Fermion systems are typical examples for non-tensor product systems. It is obvious that algebras of subsystems with mutually

disjoint regions do not mutually commute due to the anticommutativity of Fermion creation and annihilation operators and satisfy  $\mathcal{A}(\{i\}) \cap \mathcal{A}(\{j\}) = c\mathbf{1}$  ( $c \in \mathbb{C}$ ).

Our article [3] deals with the problems about joint extension of states for Fermion systems generalizing some of results in [5]. The setting of [5] is restricted to a finite-dimensional bipartite CAR system and all the results about state extensions in [5] are reduced to the special cases of those given in [3]. However, the methods of proof are different from each other and [3] relates the quantum entanglement for Fermion systems to the state extension; this is a new perspective. Therefore, before we are going to the general case in Section 5, we show some restricted results in Section 4 by using a entropy method which was obtained earlier by the author and is due to the finite-dimensionality of the systems.

## 2 The Fermion Algebra

We consider a  $\mathbf{C}^*$ -algebra  $\mathcal{A}$ , called a CAR algebra or a Fermion algebra, which is generated by its elements  $a_i$  and  $a_i^*$ ,  $i \in \mathbb{N}$  ( $\mathbb{N} = \{1, 2, \dots\}$ ) satisfying the following canonical anticommutation relations(CAR).

$$\begin{aligned} \{a_i^*, a_j\} &= \delta_{i,j} \mathbf{1} \\ \{a_i^*, a_j^*\} &= \{a_i, a_j\} = 0, \end{aligned}$$

where  $\{A, B\} = AB + BA$  (anticommutator) and  $\delta_{i,j} = 1$  for  $i = j$  and  $\delta_{i,j} = 0$  otherwise. For finite subset  $I$  of  $\mathbb{N}$ ,  $\mathcal{A}(I)$  denotes the  $\mathbf{C}^*$ -subalgebra generated by  $a_i$  and  $a_i^*$ ,  $i \in I$ .

For finite  $I$ ,  $\mathcal{A}(I)$  is known to be isomorphic to the tensor product of  $|I|$  copies of the full  $2 \times 2$  matrix algebra  $M_2(\mathbb{C})$  and hence isomorphic to  $M_{2^{|I|}}(\mathbb{C})$ . Then

$$\mathcal{A}_\infty = \bigcup_{|I| < \infty} \mathcal{A}(I)$$

has the unique  $\mathbf{C}^*$ -norm. The  $\mathbf{C}^*$  algebra  $\mathcal{A}$  together with its individual elements  $\{a_i, a_i^* | i \in \mathbb{Z}\}$  is uniquely defined up to isomorphism and is isomorphic to the UHF-algebra  $\overline{\otimes}_{i \in \mathbb{Z}} M_2(\mathbb{C})$ , where the bar denotes the norm completion.  $\mathcal{A}$  has the unique tracial state  $\tau$  as the extension of the unique tracial state of  $\mathcal{A}(I)$ ,  $|I| < \infty$ .

A crucial role is played by the unique automorphism  $\Theta$  of  $\mathcal{A}$  characterized by

$$\Theta(a_i) = -a_i, \quad \Theta(a_i^*) = -a_i^*$$

for all  $i \in \mathbb{N}$ . The even and odd parts of  $\mathcal{A}$  and  $\mathcal{A}(I)$  are defined by

$$\mathcal{A}_{\pm} \equiv \{A \in \mathcal{A} | \Theta(A) = \pm A\},$$

For any  $A \in \mathcal{A}$  (or  $\mathcal{A}(I)$ ), we have the following decomposition

$$A_{\pm} = A_+ + A_-, \quad A_{\pm} = \frac{1}{2}(A \pm \Theta(A)) \in \mathcal{A}_{\pm} \text{ (or } \mathcal{A}(I)_{\pm}).$$

A state  $\varphi$  of  $\mathcal{A}$  or  $\mathcal{A}(I)$  is called even if it is  $\Theta$ -invariant:

$$\varphi(\Theta(A)) = \varphi(A)$$

for all  $A \in \mathcal{A}$  (or  $A \in \mathcal{A}(I)$ ).

For a state  $\varphi$  of a  $\mathbf{C}^*$ -algebra  $\mathcal{A}$  ( $\mathcal{A}(I)$ ),  $\{\mathcal{H}_{\varphi}, \pi_{\varphi}, \Omega_{\varphi}\}$  denotes the GNS triplet of a Hilbert space  $\mathcal{H}_{\varphi}$ , a representation  $\pi_{\varphi}$  of  $\mathcal{A}$  (of  $\mathcal{A}(I)$ ), and a vector  $\Omega_{\varphi} \in \mathcal{H}_{\varphi}$ , which is cyclic for  $\pi_{\varphi}(\mathcal{A})$  ( $\pi_{\varphi}(\mathcal{A}(I))$ ) and satisfies

$$\varphi(A) = (\Omega_{\varphi}, \pi_{\varphi}(A)\Omega_{\varphi})$$

for all  $A \in \mathcal{A}$  ( $\mathcal{A}(I)$ ). For any  $x \in \mathcal{B}(\mathcal{H}_{\varphi})$ , we write

$$\overline{\varphi}(x) = (\Omega_{\varphi}, x\Omega_{\varphi}).$$

### 3 Product State Extension

As subsystems, we consider  $\mathcal{A}(I)$  with mutually disjoint subsets  $I$ 's. For a pair of disjoint subsets  $I_1$  and  $I_2$  of  $\mathbb{N}$ , let  $\varphi_1$  and  $\varphi_2$  be given states of  $\mathcal{A}(I_1)$  and  $\mathcal{A}(I_2)$ , respectively. If a state  $\varphi$  of the joint system  $\mathcal{A}(I_1 \cup I_2)$  (which is the same as the  $C^*$ -subalgebra of  $\mathcal{A}$  generated by  $\mathcal{A}(I_1)$  and  $\mathcal{A}(I_2)$ ) coincides with  $\varphi_1$  on  $\mathcal{A}(I_1)$  and  $\varphi_2$  on  $\mathcal{A}(I_2)$ , i.e.,

$$\begin{aligned}\varphi(A_1) &= \varphi_1(A_1), & A_1 \in \mathcal{A}(I_1), \\ \varphi(A_2) &= \varphi_2(A_2), & A_2 \in \mathcal{A}(I_2),\end{aligned}$$

then  $\varphi$  is called a joint extension of  $\varphi_1$  and  $\varphi_2$ . As a special case, if

$$\varphi(A_1 A_2) = \varphi_1(A_1) \varphi_2(A_2) \tag{4}$$

holds for all  $A_1 \in \mathcal{A}(I_1)$  and all  $A_2 \in \mathcal{A}(I_2)$ , then  $\varphi$  is called a product state extension of  $\varphi_1$  and  $\varphi_2$ . It is a simple generalization of the product state (3) to the general (i.e., not necessarily commutative) systems.

## 4 Finite-Dimensional Case

### 4.1 State Extension for the Bipartite System

We first consider a finite dimensional bipartite Fermion systems establishing the following Theorem 1 on the product property of the states with given pure marginal states. The corresponding result of this theorem can be generalized to the more general cases where the number of subsystems is arbitrary (including infinity), and each of subsystem is not necessary finite-dimensional. Nevertheless, the proof of Theorem 1 making use of von Neumann entropy cannot be generalized to the infinite-dimensional systems and may be of some interest by itself. It is

also a crucial tool for the characterization of “quantum entanglement” in Subsection 4.2.

**Theorem 1.** *Let  $\mathcal{A}(\{1\})$  and  $\mathcal{A}(\{2\})$  be a pair of Fermion systems generated by one-particle Fermions  $\{a_1, a_1^*\}$  and  $\{a_2, a_2^*\}$ , respectively. Let  $\omega$  be a state of  $\mathcal{A}$ . Suppose that its restrictions to  $\mathcal{A}(\{1\})$  and  $\mathcal{A}(\{2\})$  are both pure states. Then  $\omega$  is a pure state of  $\mathcal{A}$  and has the following product property over  $\mathcal{A}(\{1\})$  and  $\mathcal{A}(\{1\})'$ ,*

$$\omega(AB) = \omega(A)\omega(B), \quad (5)$$

*for every  $A \in \mathcal{A}(\{1\})$  and  $B \in \mathcal{A}(\{1\})'$ . The restriction of  $\omega$  to  $\mathcal{A}(\{1\})'$  is also a pure state.*

We shall state the proof of this theorem so as to explain the motivation of the present investigation. (As for the other theorems in this note, see [3].)

*Proof*

Let  $\omega_1$  be the restriction of  $\omega$  to  $\mathcal{A}(\{1\})$  and  $\omega_2$  be the restriction of  $\omega$  to  $\mathcal{A}(\{2\})$ . By the assumption that  $\omega_1$  and  $\omega_2$  are pure states, both von Neumann entropies vanish:

$$S(\omega_1) = S(\omega_2) = 0$$

The strong subadditivity property of entropy for finite-dimensional Fermion systems holds (6), the subadditivity property of entropy holds *a fortiori*.

$$S(\omega|_{\mathcal{A}}) \leq S(\omega_1) + S(\omega_2) = 0 + 0 = 0.$$

Thus the positivity of entropy implies

$$S(\omega|_{\mathcal{A}}) = 0.$$

We note that

$$\mathcal{A} = \mathcal{A}(\{1\}) \vee \mathcal{A}(\{2\}) = \mathcal{A}(\{1\}) \otimes \mathcal{A}(\{1\})'.$$

By this vanishing result of entropy of  $\omega$ , we conclude that  $\omega$  is a pure state of  $\mathcal{A}$ . Since  $\mathcal{A}$  is a full matrix algebra, every pure state is a vector state. Therefore, for this  $\omega$ , there exists a unique normalized vector  $\eta_{(\omega)}$  in  $\mathcal{H}$  up to a phase factor satisfying

$$\omega(A) = (A\eta_{(\omega)}, \eta_{(\omega)})_{\mathcal{H}}$$

for any  $A \in \mathcal{A}$ .

The product property (5) follows from the well-known Lemma IV.4.11 of [9]. By this product property and the tensor-product structure between  $\mathcal{A}(\{1\})$  and  $\mathcal{A}(\{1\})'$ , the purity of  $\omega$  implies that of the restriction of  $\omega$  to  $\mathcal{A}(\{1\})'$ .  $\square$

## 4.2 Von Neumann Entropy and Quantum Entanglement

We collect some basic properties entropy for Fermion systems. The following inequality of von Neumann entropy is called the SSA property and can be shown based on some results on the conditional expectation (see [2]). (The SSA for the tensor-product systems is shown by Lieb and Ruskai in [4].)

**Theorem 2 (SSA).** *For finite subsets  $I$  and  $J$ , the following strong subadditivity of von Neumann entropy  $S$  holds for any state  $\varphi$ :*

$$S(\varphi_{I \cup J}) - S(\varphi_I) - S(\varphi_J) + S(\varphi_{I \cap J}) \leq 0. \quad (6)$$

Let  $I$  and  $J$  be two disjoint finite regions. For tensor-product systems, the so-called “triangle inequality of entropy” holds for any state  $\varphi$  [1]

$$|S(\varphi_I) - S(\varphi_J)| \leq S(\varphi_{I \cup J}).$$

However, this inequality fails to hold for Fermion systems. The violation of the triangle inequality describes the characteristics



feature of quantum entanglement for Fermion systems which cannot exist in any tensor-product systems.

**Theorem 3.** *Let  $\mathcal{A}(\{1\})$  and  $\mathcal{A}(\{2\})$  be as Theorem 1. For any positive number  $x \in [0, \log 2]$ , there exists a pure state  $\varphi$  such that*

$$|S(\varphi|_{\mathcal{A}(\{1\})}) - S(\varphi|_{\mathcal{A}(\{2\})})| = x$$

If the above  $x$  is strictly positive, we say that the pure state  $\varphi$  has “half-sided entanglement”. (See [5] for details.)

## 5 General Case (arbitrary numbers of subsystems of arbitrary dimensions )

We go back to the problem of state extension. For an arbitrary (finite or infinite) number of subsystems,  $\mathcal{A}(I_1), \mathcal{A}(I_2), \dots$  with mutually disjoint  $I$ 's and a set of given states  $\varphi_i$  of  $\mathcal{A}(I_i)$ , a state  $\varphi$  of  $\mathcal{A}(\cup_i I_i)$  is called a product state extension if it satisfies (4) for any distinct  $i$  and  $j$ .

We give the following Lemmas.

**Lemma 1.** *For disjoint  $I_1$  and  $I_2$ , let  $\varphi$  be a state of  $\mathcal{A}(I_1 \cup I_2)$  with its restrictions  $\varphi_1$  and  $\varphi_2$  to  $\mathcal{A}(I_1)$  and  $\mathcal{A}(I_2)$ . Then the representation  $\pi_\varphi$  of  $\mathcal{A}(I_1)$  is quasi-equivalent to  $\pi_{\varphi_1} \oplus \pi_{\varphi_1\Theta}$ .*

**Lemma 2.** *If  $\pi_{\varphi_1}$  and  $\pi_{\varphi_1\Theta}$  are disjoint, then*

$$\mathcal{H}_{\varphi+} \perp \mathcal{H}_{\varphi-}, \quad (7)$$

*and  $\pi_\varphi$  restricted to  $\mathcal{H}_{\varphi\pm}$  are quasi-equivalent to  $\pi_{\varphi_1}$  and  $\pi_{\varphi_1\Theta}$ .*

We have the following Theorem.

**Theorem 4.** *Let  $I_1, I_2, \dots$  be an arbitrary (finite or infinite) number of mutually disjoint subsets of  $\mathbb{N}$  and  $\varphi_i$  be a given state*

of  $\mathcal{A}(I_i)$  for each  $i$ .

(1) A product state extension of  $\varphi_i$ ,  $i = 1, 2, \dots$ , exists if and only if all states  $\varphi_i$  except at most one are even. It is unique if it exists. It is even if and only if all  $\varphi_i$  are even.

(2) Suppose that all  $\varphi_i$  are pure. If there exists a joint extension of  $\varphi_i$ ,  $i = 1, 2, \dots$ , then all states  $\varphi_i$  except at most one have to be even. If this is the case, the joint extension is uniquely given by the product state extension and is a pure state.

*Remark.* In Theorem 4 (2), the product state property (3) is not assumed but it is derived from the purity assumption for all  $\varphi_i$ .

The purity of all  $\varphi_i$  does not follow from that of their joint extension  $\varphi$  in general. For a product state extension  $\varphi$ , however, we have the following two theorems about consequences of purity of  $\varphi$ .

**Theorem 5.** *Let  $\varphi$  be the product state extension of states  $\varphi_i$  with disjoint  $I_i$ . Assume that all  $\varphi_i$  except  $\varphi_1$  are even.*

(1)  $\varphi_1$  is pure if  $\varphi$  is pure.

(2) Assume that  $\pi_{\varphi_1}$  and  $\pi_{\varphi_1\Theta}$  are not disjoint. Then  $\varphi$  is pure if and only if all  $\varphi_i$  are pure. In particular, this is the case if  $\varphi$  is even.

*Remark.* If  $I_1$  is finite, the assumption of Theorem 5 (2) holds and hence the conclusion follows automatically.

In the case not covered by Theorem 5, the following result gives a complete analysis if we take  $\cup_{i \geq 2} I_i$  in Theorem 5 as one subset of  $\mathbb{N}$ .

**Theorem 6.** *Let  $\varphi$  be the product state extension of states  $\varphi_1$  and  $\varphi_2$  of  $\mathcal{A}(I_1)$  and  $\mathcal{A}(I_2)$  with disjoint  $I_1$  and  $I_2$  where  $\varphi_2$  is even and  $\varphi_1$  is such that  $\pi_{\varphi_1}$  and  $\pi_{\varphi_1\Theta}$  are disjoint.*

(1)  $\varphi$  is pure if and only if  $\varphi_1$  and the restriction  $\varphi_{2+}$  of  $\varphi_2$  to

$\mathcal{A}(\mathbf{I}_2)_+$  are both pure.

(2) Assume that  $\varphi$  is pure.  $\varphi_2$  is not pure if and only if

$$\varphi_2 = \frac{1}{2}(\widehat{\varphi}_2 + \widehat{\varphi}_2\Theta)$$

where  $\widehat{\varphi}_2$  is pure and  $\pi_{\widehat{\varphi}_2}$  and  $\pi_{\widehat{\varphi}_2\Theta}$  are disjoint.

*Remark.* The first two theorems are some generalization of results in [7] with the following overlap. The first part of Theorem 4 (1) is given in [7] as Theorem 5.4 (the if part and uniqueness) and a discussion after Definition 5.1 (the only if part). Theorem 4 (2) and Theorem 5 are given in Theorem 5.5 of [7] under the assumption that all  $\varphi_i$  are even.

## 6 Other State Extensions

The rest of our results concerns a joint extension of states of two subsystems, not satisfying the product state property (3). We need a few more notation. For two states  $\varphi$  and  $\psi$  of a  $\mathbf{C}^*$ -algebra  $\mathcal{A}(\mathbf{I}_1)$ , consider any representation  $\pi$  of  $\mathcal{A}(\mathbf{I}_1)$  on a Hilbert space  $\mathcal{H}$  containing vectors  $\Phi$  and  $\Psi$  such that

$$\varphi(A) = (\Phi, \pi(A)\Phi), \quad \psi(A) = (\Psi, \pi(A)\Psi).$$

The transition probability between  $\varphi$  and  $\psi$  is defined ([10]) by

$$P(\varphi, \psi) \equiv \sup |(\Phi, \Psi)|^2$$

where the supremum is taken over all  $\mathcal{H}$ ,  $\pi$ ,  $\Phi$  and  $\Psi$  as described above. For a state  $\varphi_1$  of  $\mathcal{A}(\mathbf{I}_1)$ , we need the following quantity

$$p(\varphi_1) \equiv P(\varphi_1, \varphi_1\Theta)^{1/2}$$

where  $\varphi_1\Theta$  denotes the state  $\varphi_1\Theta(A) = \varphi_1(\Theta(A))$ ,  $A \in \mathcal{A}(\mathbf{I}_1)$ .

If  $\varphi_1$  is pure, then  $\varphi_1\Theta$  is also pure and the representations  $\pi_{\varphi_1}$  and  $\pi_{\varphi_1\Theta}$  are both irreducible. There are two alternatives.

- ( $\alpha$ ) They are mutually disjoint. In this case  $p(\varphi_1) = 0$ .  
 ( $\beta$ ) They are unitarily equivalent.

In the case ( $\beta$ ), there exists a self-adjoint unitary  $u_1$  on  $\mathcal{H}_{\varphi_1}$  such that

$$\begin{aligned} u_1 \pi_{\varphi_1}(A) u_1 &= \pi_{\varphi_1}(\Theta(A)), \quad A \in \mathcal{A}(I_1), \\ (\Omega_{\varphi_1}, u_1 \Omega_{\varphi_1}) &\geq 0. \end{aligned}$$

For two states  $\varphi$  and  $\psi$ , we introduce

$$\lambda(\varphi, \psi) \equiv \sup\{\lambda \in \mathbb{R}; \varphi - \lambda\psi \geq 0\}$$

Since  $\varphi - \lambda_n \psi \geq 0$  and  $\lim \lambda_n = \lambda$  imply  $\varphi - \lambda\psi \geq 0$ , we have

$$\varphi \geq \lambda(\varphi, \psi)\psi.$$

We need

$$\lambda(\varphi_2) \equiv \lambda(\varphi_2, \varphi_2 \Theta).$$

The next Theorem provides a complete answer for a joint extension  $\varphi$  of states  $\varphi_1$  and  $\varphi_2$  of  $\mathcal{A}(I_1)$  and  $\mathcal{A}(I_2)$ , when one of them is pure.

**Theorem 7.** *Let  $\varphi_1$  and  $\varphi_2$  be states of  $\mathcal{A}(I_1)$  and  $\mathcal{A}(I_2)$  for disjoint subsets  $I_1$  and  $I_2$ . Assume that  $\varphi_1$  is pure.*

(1) *A joint extension  $\varphi$  of  $\varphi_1$  and  $\varphi_2$  exists if and only if*

$$\lambda(\varphi_2) \geq \frac{1 - p(\varphi_1)}{1 + p(\varphi_1)}. \quad (8)$$

(2) *If eq. (8) holds and if  $p(\varphi_1) \neq 0$ , then a joint extension  $\varphi$  is unique and satisfies*

$$\begin{aligned} \varphi(A_1 A_2) &= \varphi_1(A_1) \varphi_2(A_{2+}) + \frac{1}{p(\varphi_1)} f(A_1) \varphi_2(A_{2-}), \\ f(A_1) &\equiv \overline{\varphi_1}(\pi_{\varphi_1}(A_1) u_1) \end{aligned}$$

for  $A_1 \in \mathcal{A}(I_1)$  and  $A_2 = A_{2+} + A_{2-}$ ,  $A_{2\pm} \in \mathcal{A}(I_2)_{\pm}$ .

(3) If  $p(\varphi_1) = 0$ , (8) is equivalent to evenness of  $\varphi_2$ . If this is the case, at least a product state extension of Theorem 4 exists.

(4) Assume that  $p(\varphi_1) = 0$  and  $\varphi_2$  is even. There exists a joint extension of  $\varphi_1$  and  $\varphi_2$  other than the unique product state extension if and only if  $\varphi_1$  and  $\varphi_2$  satisfy the following pair of conditions:

(4-i)  $\pi_{\varphi_1}$  and  $\pi_{\varphi_1\Theta}$  are unitarily equivalent.

(4-ii) There exists a state  $\tilde{\varphi}_2$  of  $\mathcal{A}(I_2)$  such that  $\tilde{\varphi}_2 \neq \tilde{\varphi}_2\Theta$  and

$$\varphi_2 = \frac{1}{2}(\tilde{\varphi}_2 + \tilde{\varphi}_2\Theta).$$

(5) If  $p(\varphi_1) = 0$ , then corresponding to each  $\tilde{\varphi}_2$  above, there exists a joint extension  $\varphi$  which satisfies

$$\varphi(A_1 A_2) = \varphi_1(A_1) \varphi_2(A_{2+}) + \overline{\varphi_1}(\pi_{\varphi_1}(A_1) u_1) \tilde{\varphi}_2(A_{2-}). \quad (9)$$

Such extensions along with the unique product state extension (which satisfies eq. (9) for  $\tilde{\varphi}_2 = \varphi_2$ ) exhaust all joint extensions of  $\varphi_1$  and  $\varphi_2$  when  $p(\varphi_1) = 0$ .

*Remark.* The eq.(8) is sufficient for the existence of a joint extension also for general states  $\varphi_1$  and  $\varphi_2$ .

We have a necessary and sufficient condition for the existence of a joint extension of states  $\varphi_1$  and  $\varphi_2$  under a specific condition on  $\varphi_1$ .

**Theorem 8.** Let  $\varphi_1$  and  $\varphi_2$  be states of  $\mathcal{A}(I_1)$  and  $\mathcal{A}(I_2)$  for disjoint subsets  $I_1$  and  $I_2$ . Assume that  $\pi_{\varphi_1}$  and  $\pi_{\varphi_1\Theta}$  are disjoint. Then a joint extension of  $\varphi_1$  and  $\varphi_2$  exists if and only if  $\varphi_2$  is even.

## 7 Examples

### Example 1

Let  $I_1$  and  $I_2$  be mutually disjoint finite subsets of  $\mathbb{N}$ . Let  $\varrho \in$

$\mathcal{A}(I_1 \cup I_2)$  be an invertible density matrix, namely  $\varrho \geq \lambda \mathbf{1}$  for some  $\lambda > 0$  and  $\text{Tr}(\varrho) = 1$ , where  $\text{Tr}$  denotes the matrix trace on  $\mathcal{A}(I_1 \cup I_2)$ . Take any  $x = x^* \in \mathcal{A}(I_1)_-$  and  $y = y^* \in \mathcal{A}(I_2)_-$  satisfying  $\|x\|\|y\| \leq \lambda$ . Let  $\varphi_1(A_1) \equiv \text{Tr}(\varrho A_1)$  for  $A_1 \in \mathcal{A}(I_1)$  and  $\varphi_2(A_2) \equiv \text{Tr}(\varrho A_2)$  for  $A_2 \in \mathcal{A}(I_2)$ . Then

$$\varphi'_\varrho(A) \equiv \text{Tr}(\varrho' A), \quad \varrho' \equiv \varrho + ixy.$$

for  $A \in \mathcal{A}(I_1 \cup I_2)$  is a state of  $\mathcal{A}(I_1 \cup I_2)$  and has  $\varphi_1$  and  $\varphi_2$  as its restrictions to  $\mathcal{A}(I_1)$  and  $\mathcal{A}(I_2)$ , irrespective of the choice of  $x$  and  $y$  satisfying the above conditions.

### Example 2

Let  $I_1$  and  $I_2$  be mutually disjoint subsets of  $\mathbb{N}$ . Let  $\varphi$  and  $\psi$  be states of  $\mathcal{A}(I_1)$  and  $\mathcal{A}(I_2)$  such that

$$\varphi = \sum_i \lambda_i \varphi_i, \quad \psi = \sum_i \lambda_i \psi_i, \quad (0 < \lambda_i, \sum_i \lambda_i = 1),$$

where  $\varphi_i$  and  $\psi_i$  are states of  $\mathcal{A}(I_1)$  and  $\mathcal{A}(I_2)$  which have a joint extension  $\chi_i$  for each  $i$ .

$$\chi = \sum_i \lambda_i \chi_i$$

is a joint extension of  $\varphi$  and  $\psi$ .

This simple example yields next more elaborate ones.

### Example 3

Let  $\varphi$  and  $\psi$  be states of  $\mathcal{A}(I_1)$  and  $\mathcal{A}(I_2)$  for disjoint  $I_1$  and  $I_2$  with (non-trivial) decompositions

$$\varphi = \lambda \varphi_1 + (1 - \lambda) \varphi_2, \quad \psi = \mu \psi_1 + (1 - \mu) \psi_2, \quad (0 < \lambda, \mu < 1)$$

where  $\varphi_1$  and  $\varphi_2$  are even. Product state extensions  $\varphi_i \psi_j$  of  $\varphi_i$  and  $\psi_j$  yield

$$\begin{aligned} \chi \equiv & (\lambda\mu + \kappa) \varphi_1 \psi_1 + (\lambda(1 - \mu) - \kappa) \varphi_1 \psi_2 \\ & ((1 - \lambda)\mu - \kappa) \varphi_2 \psi_1 + ((1 - \lambda)(1 - \mu) + \kappa) \varphi_2 \psi_2, \end{aligned}$$

which is a joint extension of  $\varphi$  and  $\psi$  for all  $\kappa \in \mathbb{R}$  satisfying

$$-\min(\lambda\mu, (1-\lambda)(1-\mu)) \leq \kappa \leq \min((1-\lambda)\mu, \lambda(1-\mu)).$$

Example 4

Let  $\varphi_k, k = 1, \dots, m$  and  $\psi_l, l = 1, \dots, n$  be states of  $\mathcal{A}(I_1)$  and  $\mathcal{A}(I_2)$  for disjoint  $I_1$  and  $I_2$ . Let

$$\varphi = \sum_{k=1}^m \lambda_k \varphi_k, \quad \psi = \sum_{l=1}^n \mu_l \psi_l$$

with  $\lambda_k, \mu_l > 0, \sum \lambda_k = \sum \mu_l = 1$ . Assume that there exists a joint extension  $\chi_{kl}$  of  $\varphi_k$  and  $\psi_l$  for each  $k$  and  $l$ . Then

$$\chi = \sum_{kl} (\lambda_k \mu_l + \kappa_{kl}) \chi_{kl} \tag{10}$$

is a joint extension if

$$(\lambda_k \mu_l + \kappa_{kl}) \geq 0, \quad \sum_l \kappa_{kl} = \sum_k \kappa_{kl} = 0.$$

Since the constraint for  $mn$  parameters  $\{\kappa_{kl}\}$  are effectively  $m + n - 1$  linear relations (because  $\sum_{kl} \kappa_{kl} = 0$  is common for  $\sum_l \kappa_{kl} = 0$  and  $\sum_k \kappa_{kl} = 0$ ), we have  $mn - (m + n - 1) = (m - 1)(n - 1)$  parameters for the joint extension (10).

## References

- [1] H.Araki and E.H. Lieb, Entropy inequalities, *Commun. Math. Phys.* **18**,(1970), 160-170.
- [2] H.Araki and H.Moriya, Equilibrium Statistical Mechanics of Fermion Lattice Systems, *preprint*.
- [3] H.Araki and H.Moriya, State Extension from Subsystems to the Jonint System. *preprint*.

- [4] E.H.Lieb and M.B.Ruskai, Proof of the strong subadditivity of quantum-mechanical entropy, *J. Math. Phys.* **14**(1973), 1938-1941.
- [5] H.Moriya, Some aspects of quantum entanglement for CAR systems, *Lett. Math. Phys.* **60**(2002), 109-121.
- [6] R.T.Powers, Representations of the canonical anticommutation relations, Thesis, Princeton University(1967).
- [7] M.Takesaki, *Theory of Operator Algebras I*, Springer, 1979.
- [8] A.Uhlmann, The “transition probability” in the state space of a  $*$ -algebra. *Rep. Math. Phys.* **9**(1976), 273-277.